Statistics of forced thermally activated escape events out of a metastable state: Most probable escape force and escape-force moments

Mykhaylo Evstigneev*
Fakultät für Physik, Universität Bielefeld, 33615, Bielefeld, Germany
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The dynamics of a number of experimental systems can be described as thermally activated escape out of a metastable state over a potential barrier, whose height is being constantly reduced in time by an increasing external force. In such systems, one can distinguish two loading regimes: for slow loading, the distribution of the force values at which escape occurs is a monotonically decreasing function, while for fast loading, the escape-force distribution has a maximum at some nonzero force value. In this work, an approximate relation between the most probable escape force and the first two moments thereof is derived for fast loading, and the expression for the first two force moments vs loading rate is obtained for slow loading. Then, for a special but physically well-motivated functional form of the escape rate, the most probable escape force is found analytically as a function of the loading rate. The high accuracy of these expressions is confirmed by comparing them with numerical results for realistic parameter values.

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I. INTRODUCTION

Thermally activated escape out of a potential minimum is of central importance in a number of experimental studies, including determination of bond strength of biocomplexes [1–8], friction at the atomic scale [9–11], magnetization reversal in nanomagnets [12,13], and flux jumps in Josephson junctions [14], to name but a few. All these systems possess one or many metastable states, which the system leaves due to a thermally activated transition over some potential barrier. Experimentally, one can measure the force-dependent off rate of this process, i.e., the probability to leave the metastable state per unit time. The well-known Kramers-Arrhenius law [see Eq. (3) below] allows one to learn a great deal about the system's characteristic time and energy scales from the knowledge of the off rate.

In a typical experiment, one applies an (approximately) linearly increasing force to constantly reduce the height of the potential barrier, thus increasing the off rate, until a thermally activated jump out of the metastable state occurs. Upon repeating such trials many times, one obtains a distribution of the escape forces, whose most important characteristics are the most probable and the mean escape force, as well as the variance of escape forces. Then, based on this information, one attempts to reconstruct the off rate, and thus to characterize the system of interest.

In this paper, a relation between the first two moments of the escape force and the most probable force at the moment of escape is derived. Then, a simple ansatz for the off-rate is introduced and motivated physically. For this functional form of the off rate, the most probable escape force is found analytically, allowing one to obtain the first two force moments as well. It is shown that the results presented in this paper for the first two moments of the escape force are complementary to those obtained by Garg [15] some 13 years ago.

*mykhaylo@physik.uni-bielefeld.de

II. RATE EQUATION

The main ingredients of the rate description are the probability $p(t)$ that the system will remain in the same metastable state at the moment of time t, and the off rate $\omega[f(t)]$ to leave that state per unit time. The argument of the off rate is the acting force $f(t)$, which reduces the potential barrier and thus increases the probability of escape. The temporal evolution of the survival probability is governed by the following single-step rate equation:

$$\dot{p}(t) = -\omega[f(t)]p(t), \tag{1}$$

where the acting force grows linearly as

$$f(t) = rt, \tag{2}$$

r being the loading rate.

The off rate is given by the Kramers-Arrhenius law [16,17]

$$\omega(f) = \Omega(f)e^{-\Delta V(f)/k_B T}, \tag{3}$$

where $\Delta V(f)$ is the force-dependent height of the potential barrier, which needs to be overcome to leave the metastable state, $\Omega(f)$ is the attempt frequency, which depends subexponentially on the acting force, and $k_B T$ is the thermal energy.

With the change of variables (2), we find from the rate equation (1) that the survival probability in the force domain decays according to

$$p'(f) = -\frac{1}{r}\omega(f)p(f). \tag{4}$$

The solution of this equation reads

$$p(f) = \exp\left(-\frac{1}{r}\int_0^f df'\,\omega(f')\right), \tag{5}$$

as can be verified by differentiation.

The simple single-step rate description (1) of the system's dynamics is an approximation. It is valid, provided that the time scale of thermally activated escape is much slower than the time scale of all other (microscopic) degrees of freedom. This time-scale separation condition implies that the barrier height must be much larger than the thermal energy whenever the system finds itself in the metastable state, i.e., before the acting force reaches the value $\hat{f}$ such that

$$\Delta V(\hat{f}) = Ak_BT, \tag{6}$$

where A is some number greater than 1, say, $A=5$. The rate equation is valid if all escape events occur with overwhelming probability before the force reaches the value $\hat{f}$. In other words, the survival probability at this force value must be sufficiently small,

$$p(\hat{f}) < \epsilon, \tag{7}$$

where ϵ is some small number, say, $\epsilon=10^{-3}$. Incorporating the result (5), we find that the rate approach is valid for loading rates smaller than the value

$$r < \frac{1}{\ln \epsilon^{-1}} \int_0^{\hat{f}} df\, \omega(f). \tag{8}$$

We note that, in view of the exponentially increasing character of the force-dependent off rate (3), the dominant contribution to the integral comes from the force region just below $\hat{f}$. This allows us to simplify the integral by approximating the off rate in the vicinity of $\hat{f}$ as

$$\omega(f) \approx \omega(\hat{f}) e^{\lambda(\hat{f})(f-\hat{f})}, \tag{9}$$

where

$$\lambda(f) \coloneqq \frac{d \ln \omega(f)}{df}. \tag{10}$$

With the approximation (9), the condition of validity of the rate equation takes the form

$$r < \frac{1}{\ln \epsilon^{-1}} \frac{\omega(\hat{f})}{\lambda(\hat{f})}. \tag{11}$$

III. RELATION BETWEEN THE FIRST TWO FORCE MOMENTS AND THE MOST PROBABLE ESCAPE FORCE

In many experiments listed in the beginning of the Introduction, it is customary to measure the most probable force at which the distribution of escape forces, $-p'(f)$, is maximized. Since the maximum f_* of this distribution is found from the condition $p''(f_*)=0$, differentiation of Eq. (4) yields the equation from which f_* can be found:

$$\omega'(f_*) = \frac{\omega^2(f_*)}{r}. \tag{12}$$

Formally, the solution of this equation may assume both positive (at high loading rates r) and negative (at low r) values. The critical loading rate separating these two regimes can be found by setting f_* in Eq. (12) to zero:

$$r_c = \frac{\omega^2(0)}{\omega'(0)}. \tag{13}$$

Experimentally, the applied force usually starts to increase from the initial value $f=0$, so that negative force values are not realized in practice. This means that for small loading rates, $r<r_c$, the distribution of escape forces is a monotonically decreasing function having a maximum at the initial force value:

$$f_* = 0 \quad \text{for } r < r_c. \tag{14}$$

In the following, we assume that $r>r_c$, so that $f_*>0$.

Given the survival probability (5), one can find the force moments as

$$\langle f^n \rangle = -\int_0^{f_c} df\, f^n p'(f), \tag{15}$$

where the upper limit of integration equals that critical force at which the barrier vanishes. The analytical relation between f_* and force moments follows immediately from Eq. (15), where one uses the explicit expression (5) for the survival probability, and replaces the loading rate r with the equivalent expression $\omega^2(f_*)/\omega'(f_*)$ from Eq. (12). The resulting relation involves a double integral and may be time consuming to calculate numerically.

To simplify this relation, we note that the largest contribution to the integral (15) comes from the force region around the most probable force value f_*. This observation suggests that, to evaluate this integral, it is advantageous to expand the logarithm of the off rate around f_* [cf. Eq. (9)],

$$\omega(f) \approx \omega(f_*) e^{\lambda(f_*)(f-f_*)}, \tag{16}$$

where $\lambda(f)$ is defined in Eq. (10). This approximation results in the following expression for the survival probability:

$$\begin{aligned} p(f) &\approx \exp\left(- \frac{\omega(f_*) e^{-\lambda(f_*)f_*}(e^{\lambda(f_*)f} - 1)}{r\lambda(f_*)} \right) \\ &= \exp[e^{-\lambda(f_*)f_*}(1 - e^{\lambda(f_*)f})], \end{aligned} \tag{17}$$

where we used the relation (12) and the definition of $\lambda(f)$ to obtain the second equality. The approximation (17) has the correct behavior at low forces, where $p(f)$ assumes the value 1, at high forces, where it approaches 0, and in the transition region around f_* between these two extremes. Furthermore, in view of the fact that the survival probability (5) drops to zero well before the force reaches the critical value (see the discussion at the end of the previous section), we can use the approximation (17) in the expression (15) and set the upper limit of integration to infinity. Then, the first two moments of the escape force are found after some algebra involving change of variables of integration:

$$\langle f \rangle \approx \frac{1}{\lambda(f_*)} e^x E_1(x), \quad \langle f^2 \rangle \approx \frac{2}{\lambda^2(f_*)} e^x G(x),$$

$$x := e^{-\lambda(f_*)f_*}, \quad r > r_c. \tag{18}$$

Here, the special function

$$E_1(x) := \int_1^\infty dz \frac{e^{-xz}}{z} = -\ln(x) - \gamma - \sum_{n=1}^{\infty} \frac{(-x)^n}{n n!} \tag{19}$$

is the exponential integral, which can be evaluated numerically using an efficient algorithm from [18], and $\gamma = 0.577\,215\,664\,9\ldots$ is the Euler-Mascheroni constant. The auxiliary special function $G(x)$ is defined as

$$G(x) := \int_1^\infty dz \frac{\ln(z)}{z} e^{-xz} = \theta + \ln(x)\left(\frac{1}{2}\ln(x) + \gamma\right) + \sum_{n=1}^{\infty} \frac{(-x)^n}{n^2 n!}, \tag{20}$$

where the constant $\theta = 0.989\,048\,872\,2\ldots$. The derivation of the second part of this identity, which allows one to evaluate the function $G(x)$ quickly and accurately, is given in the Appendix.

The quantities related by Eq. (18)—the most probable escape force and the first two moments thereof—can be determined experimentally, and therefore Eq. (18) can be employed to find the important characteristic of the system, $\lambda(f)$. Alternatively, one may wish to fit the experimental relation between the force moments $\langle f^n \rangle$, $n=1,2$, and the loading rate r based on some specific model for the off rate $\omega(f)$. Equation (18) can be used for this purpose for an arbitrary such functional form, provided the off rate increases exponentially strongly with the force [cf. Eq. (3)]. To do this, one has to employ the following parametric procedure: for each fixed most probable escape force f_*, one should find the corresponding loading rate from Eq. (12), $r = \omega^2(f_*)/\omega'(f_*)$, at which this value of f_* is realized; then, substitution of the same value of f_* into Eq. (18) yields the first two force moments corresponding to this loading rate value.

Instead of this parametric procedure, it is possible to express $\langle f \rangle$ and $\langle f^2 \rangle$ directly in terms of r in two cases. First, in the slow-pulling regime specified by Eqs. (13) and (14), the distribution (4) of escape forces is a monotonically decreasing function. Hence, the largest contribution to the integral (15) comes from the force region around $f_* = 0$, and, to evaluate the integral, one should expand the logarithm of the off rate around this force value. By following exactly the same procedure as before, we find the expressions for the first two force moments as functions of the loading rate:

$$\langle f \rangle \approx \frac{1}{\lambda(0)} e^{r_c/r} E_1(r_c/r), \quad \langle f^2 \rangle \approx \frac{2}{\lambda^2(0)} e^{r_c/r} G(r_c/r),$$

$$r \leq r_c. \tag{21}$$

The second interesting case is related to a special functional choice of the off rate. This rate ansatz allows one to solve Eq. (12) with respect to f_*; substitution of the function $f_*(r)$ thus found into Eq. (18) yields $\langle f^n \rangle$ vs r for $n=1,2$. This specific rate ansatz is introduced and motivated in the next section.

IV. RATE ANSATZ

The force-dependent barrier height $\Delta V(f)$ can be established from knowledge of the system's energy $V(x;f)$ as a function of the reaction coordinate x at a given value of the applied force. The reaction coordinate assumes a different physical interpretation in each experimental situation. For instance, it can be identified with the ligand-receptor separation in dynamic force spectroscopic experiments [1–8], with nanomagnet magnetization in magnetization reversal studies [12,13], etc. In the simplest case, one can decompose the system's energy as $V(x;f) = V(x) - fx$, so the barrier height can be found as

$$\Delta V(f) = V(x_{\max}(f)) - V(x_{\min}(f)) - f[x_{\max}(f) - x_{\min}(f)], \tag{22}$$

where $x_{\min}(f) < x_{\max}(f)$ are the positions of the extrema of the potential $V(x;f)$ for a given force f, such that $V'(x_{\min,\max}(f)) - f = 0$.

A recent numerical observation of Husson and Pincet [3] suggests that in order to characterize $\Delta V(f)$ for any potential landscape with reasonable accuracy, three parameters are sufficient. Specifically, Husson and Pincet generated 8787 energy landscapes $V(x)$ of various shapes, but with fixed force-free barrier height $\Delta V_0 \equiv \Delta V(0)$ and fixed dissociation length $\Delta x := x_{\max}(0) - x_{\min}(0)$. We note that the critical force at which the barrier disappears is related to these two quantities: $\Delta V(f_c) = 0$ at $f_c \propto \Delta V_0/\Delta x$. For each such potential tested, Husson and Pincet calculated the value of the most probable escape force f_* and the derivative $s := df_*/d(\ln r)$ at some fixed loading rate value. In this way, to each potential landscape tested, a corresponding point in the s-f_* plane was generated. An interesting finding of these authors was that, instead of being scattered more or less uniformly in the s-f_* plane, all such points fell onto a single curve, which could be accurately approximated with a parabola.

This finding has the following implication. The force-dependent barrier height $\Delta V(f)$ can be parametrized using the force-free value ΔV_0, the critical force f_c at which it disappears, and additional parameters $\alpha_0, \alpha_1, \alpha_2, \ldots$ describing the manner in which the barrier height decreases from the value ΔV_0 at $f=0$ to zero at $f=f_c$. How many such additional parameters are necessary to describe the force-dependent barrier height for a given potential landscape? To each choice $\{\alpha_i\}$ there corresponds a point in the s-f_* plane, and variations of $\{\alpha_i\}$ lead to a displacement of this point. Since for all potential landscapes tested by Husson and Pincet [3] this point happened to lie on a one-dimensional manifold—a parabolic curve—we conclude that a single additional parameter α is sufficient to describe the force-dependent barrier height $\Delta V(f)$ with good accuracy; it is this parameter which controls the position of the point on the "universal" curve from [3].

With this in mind, we propose that the energy barrier decreases with the acting force according to the power law [11]

$$\Delta V(f) = \Delta V_0 \left(1 - \frac{f}{f_c}\right)^{\alpha}. \tag{23}$$

This functional form can be regarded as the first-order term of the self-similar factor approximation [19] to the true force-dependent barrier height.

For realistic potentials, we can restrict ourselves to the values of the exponent $\alpha \geqslant 1$. This statement is substantiated by the following argument [20]. By differentiating Eq. (22) with respect to the force, one can verify that the second derivative of the barrier height with respect to the acting force equals minus the force-derivative of the instantaneous distance between maximum and minimum: $\Delta V''(f) = -(d/df)[x_{\max}(f) - x_{\min}(f)]$. Since increasing the force brings the potential extrema closer to each other, the latter derivative has a negative value, meaning that $\Delta V''(f) \geqslant 0$. In other words, $\Delta V(f)$ is a convex function, so that, indeed, $\alpha \geqslant 1$ in Eq. (23).

The value of the exponent α depends on the specific approximation for the potential $V(x)$. If the potential well is sufficiently deep and its maximum sufficiently sharp, one can linearize the barrier height $\Delta V(f)$ with respect to the force, resulting in Bell's approximation [4] $\alpha \approx 1$; the exponent α becomes exactly 1 [corresponding to $\Delta V''(f)=0$] for a potential $V(x)$, which increases linearly between $x_{\min}$ and $x_{\max}$, going to $+\infty$ for $x < x_{\min}$ and to $-\infty$ for $x > x_{\max}$. On the other hand, expanding the potential $V(x)$ to the third order [6,9] yields the exponent $\alpha = 3/2$, and parabolic approximation of the potential around its extrema [3,7] gives the value $\alpha = 2$.

With respect to the prefactor Ω, generally speaking, it depends on the applied force [8,15–17]. However, we will focus on the special case of constant Ω. This assumption does not restrict the generality of the treatment for two reasons. First, by Kramers law, the effect of the force dependence of the barrier height is exponentially stronger than that of the prefactor, so that for a number of practical purposes [1–3,10–13], the prefactor can indeed be taken constant. More important, if the prefactor is a function of the acting force, $\Omega = \Omega(f)$, then one can replace it with some constant value, e.g., with $\Omega(0)$, and simultaneously subtract the function $k_B T \ln[\Omega(f)/\Omega(0)]$ from the force-dependent barrier height; evidently, this renormalization of the rate parameters will leave the off rate, and hence the statistics of the escape events, unchanged. Therefore, up to logarithmic corrections in $\Delta V(f)$, the dependence of Ω on the applied force can indeed be neglected.

These approximations result in the following rate ansatz:

$$\omega(f) = \Omega \exp\left[-\frac{\Delta V_0}{k_B T}\left(1 - \frac{f}{f_c}\right)^{\alpha}\right]. \tag{24}$$

This ansatz has been introduced (but without the physical motivation above) in the work [11]; its variants with some specifically chosen values of the exponent α have been applied to various physical systems in Refs. [1,3,4,10,12,13]. We note that with this ansatz the solution (5) of the rate Eq. (4) reads

$$p(f) = \exp\left\{-\frac{\Omega f_c}{\alpha r}\left(\frac{k_B T}{\Delta V_0}\right)^{1/\alpha} \times \left[\Gamma\left(\frac{1}{\alpha}, \frac{\Delta V_0}{k_B T}\right) - \Gamma\left(\frac{1}{\alpha}, \frac{\Delta V_0}{k_B T}\left(1 - \frac{f}{f_c}\right)^{\alpha}\right)\right]\right\}, \tag{25}$$

where the incomplete Γ-function is given by $\Gamma(a,x) := \int_0^x dt\, t^{a-1} e^{-t}$.

V. MOST PROBABLE ESCAPE FORCE

At $r > r_c$, it is convenient to rewrite Eq. (12) with the help of the ansatz (24) as

$$y e^{y} = \frac{\alpha}{\alpha - 1}\left(\frac{\Omega f_c}{\alpha r}\right)^{\alpha/(\alpha-1)}\left(\frac{k_B T}{\Delta V_0}\right)^{1/(\alpha-1)}$$

$$\text{with} \quad y := \frac{\alpha}{\alpha - 1}\frac{\Delta V_0}{k_B T}\left(1 - \frac{f_*}{f_c}\right)^{\alpha}. \tag{26}$$

Equation (26) can be solved with the help of the Lambert W function [21] defined by

$$W(x) e^{W(x)} = x. \tag{27}$$

After some algebra, we find the most probable rupture force for arbitrary α:

$$\frac{f_*}{f_c} = 1 - \left\{\frac{\alpha - 1}{\alpha}\frac{k_B T}{\Delta V_0} W\left[\frac{\alpha}{\alpha - 1}\left(\frac{\Omega f_c}{\alpha r}\right)^{\alpha/(\alpha-1)} \times \left(\frac{k_B T}{\Delta V_0}\right)^{1/(\alpha-1)}\right]\right\}^{1/\alpha}. \tag{28}$$

For the special case $\alpha = 2$, this result has recently been derived by Husson and Pincet [3].

The Lambert W function (27) can be evaluated numerically using a very efficient iterative scheme from Ref. [21]. Alternatively, one can observe that Eq. (27) implies the following continued-fraction-like representation for the W function:

$$W(x) = \ln \frac{x}{\ln \dfrac{x}{\ddots \ln x}}. \tag{29}$$

For all practical purposes, it is sufficient to truncate this approximation at the second term, i.e., to take $W(x) \approx \ln[x/\ln(x/\ln x)]$. This approximation reproduces the W function to better than a few percent accuracy for $x \geqslant 4$. Taking the value of the argument x from Eq. (28), this means that the loading rate must be smaller than some value, viz.,

$$r \leqslant \left(\frac{\alpha}{4(\alpha - 1)}\right)^{(\alpha-1)/\alpha}\left(\frac{k_B T}{\Delta V_0}\right)^{1/\alpha}\frac{f_c \Omega}{\alpha}.$$

For typical parameter values $\Omega \sim 10^6\ \text{s}^{-1}$, $f_c \sim 100$ pN, and $\Delta V_0 / k_B T \sim 100$, this means loading rates smaller than 10^6 pN/s, well within the experimentally accessible range.

In the limit $\alpha \to 1$, which is equivalent to Bell's approximation for the off rate [4], one can replace the W function in Eq. (28) with the natural logarithm, resulting in the most probable force

$$f_* = f_c\left(1 - \frac{k_B T}{\Delta V_0}\ln\frac{\Omega f_c k_B T}{r\Delta V_0}\right).$$

This expression has been used extensively in previous studies.

VI. NUMERICAL RESULTS

Next, we compare our approximations (21) and (18) combined with the exact result (28) for the rate ansatz (24) with the results obtained from the numerical integration of the expressions (15) and (25) for $n=1,2$. Furthermore, we compare the accuracy of Eqs. (18) and (21) with the corresponding approximations for $\langle f\rangle$ and $\langle f^2\rangle$ derived by Garg [15] for the case of a force-dependent prefactor Ω; for the special case of a constant prefactor Ω in the rate ansatz (24), Garg's results are

$$\frac{\langle f\rangle}{f_c} \approx 1 - \left(\frac{k_B T}{\Delta V_0}\ln X\right)^{1/\alpha}\left(1 + \frac{Y}{\alpha \ln X}\right),$$

$$\frac{\langle (f-\langle f\rangle)^2\rangle}{f_c^2} \approx \left(\frac{k_B T}{\Delta V_0}\right)^{2/\alpha}(\ln X)^{2/\alpha-2} \times\left\{\frac{\pi^2}{6\alpha^2} + \frac{1-\alpha}{\alpha^3 \ln X}\left[2\left(Y^2+Y+\frac{\pi^2}{6}\right) + \left(\frac{\pi^2 Y^2}{3} - \psi''(1)\right)\right]\right\},$$

$$X := \frac{f_c\Omega}{\alpha r}\left(\frac{k_B T}{\Delta V_0}\right)^{1/\alpha},$$

$$Y := \frac{1-\alpha}{\alpha}\ln\ln X + \gamma, \qquad (30)$$

where the value of the tetragamma function $\psi''(1) = -2.404\ldots$.

The nature of our approximation is such that the expressions (18) become exact in the Bell limit $\alpha=1$; in particular, the result for $\langle f\rangle$ coincides with the previously obtained one [5]. At the same time, Garg's result (30) for the average escape force in this limit coincides with the corresponding expression (18), where the exponential integral is approximated as $E_1(x) \approx -\ln x - \gamma$ [cf. Eq. (19)]. With respect to the variance, Garg's approximation (30) predicts a constant value independent of the loading rate for $\alpha=1$, whereas the result (18), which is exact in this limit, yields an approximately logarithmic increase of σ with r.

For a numerical test, we have chosen the exponent $\alpha = 3/2$, rate prefactor $\Omega = 10^6\ \mathrm{s}^{-1}$, and critical force $f_c = 100$ pN. Figure 1 shows the results of the numerical evaluation of the integral of Eq. (15) (solid lines), approximations (18) and (21) (dashed lines), and Garg's approximation (30) (dotted lines) for different values of the force-free barrier height ΔV_0. The calculations have been performed for pulling-rate values within the range of validity of the rate approach [see Eq. (11)].

Both approximations, the present one from Eqs. (18) and (21) and the one from Ref. [15] [see Eq. (30)], correctly

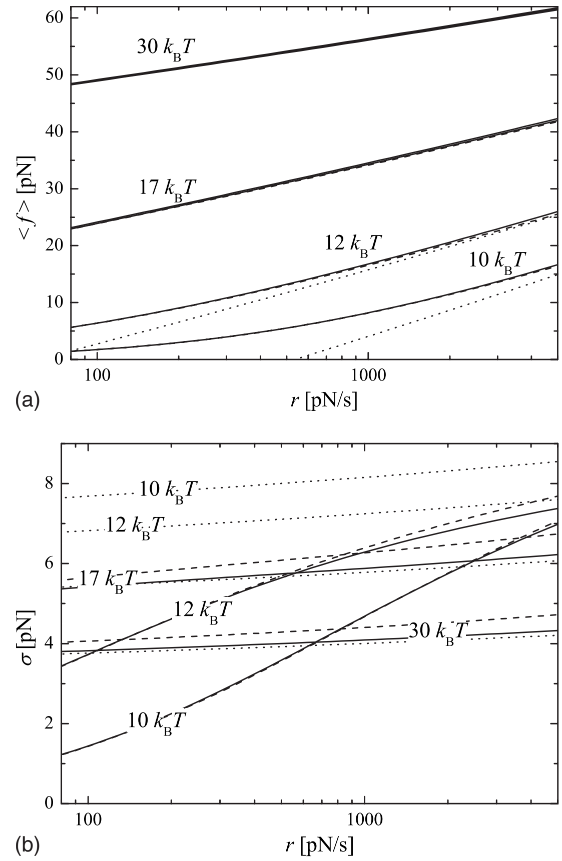

FIG. 1. (a) Average escape force and (b) dispersion of escape forces, $\sigma = \sqrt{\langle (f-\langle f\rangle)^2\rangle}$, for the rate ansatz (24) with the rate prefactor $\Omega = 10^6\ \mathrm{s}^{-1}$, critical force $f_c = 100$ pN, exponent $\alpha = 3/2$, and force-free barrier height ΔV_0 ranging from $10k_B T$ to $30k_B T$, as indicated at each curve. Solid lines are obtained by means of numerical integration of Eq. (15). Dashed lines: approximations (18) and (21). Dotted lines: approximation (30) introduced by Garg [15].

reproduce the behavior of the average force $\langle f\rangle$ with the pulling rate at large barriers; see Fig. 1(a) showing an approximately linear increase of $\langle f\rangle$ with $\ln r$. At the same time, for low barriers, Garg's approximation underestimates the average force quite notably. In contrast, the expressions (18) and (21) reproduce the numerical results correctly also in this range.

With respect to the variance of the escape forces, $\sigma = \sqrt{\langle (f-\langle f\rangle)^2\rangle}$, Fig. 1(b), our approximation overestimates it by about 10% at high barriers, whereas Garg's expression (30) yields very accurate values for large ΔV_0. The situation is quite the opposite at low barriers, where our result (18) and (21) practically coincides with the numerics, but Eq. (30) severely overestimates the variance of escape forces. We see that the approximate expressions (18), (21), and (30) can be regarded as complementary to each other in the sense that they work best in opposite ranges of barrier height.

VII. CONCLUSIONS

In experiments involving the forced escape of the system of interest out of a metastable state [1–15], one can distinguish two regimes of loading. For loading rates smaller than the value defined by Eq. (13), the experimentally observed escape-force distribution (4) is a monotonically decreasing function with the maximum at $f_*=0$; for loading rates higher than this value, it has a maximum at positive escape force.

The main results of this work are as follows. The expressions for the first two moments of the escape force as functions of loading rate are obtained in the slow-loading regime [see Eq. (21)], and the relation between these quantities and the most probable escape force is found for fast loading [see Eq. (18)]. Furthermore, for the special but physically well-motivated functional form of the off rate (24), the most probable escape force (28) is found analytically, allowing one to obtain also the first two force moments as functions of the loading rate for this choice of the off rate. If one wishes to study the force moments for the off rates of the functional form different from (24), one can still use the results of this paper as follows. While in the slow-loading regime, the expression (21) remains valid for any functional form of the off rate (provided it increases exponentially with the force), in the regime of fast loading, one can use the most probable force as a parameter: for any value of $f_*>0$, one can calculate the force moments using Eq. (18) and then find the corresponding loading rate at which these values are realized from Eq. (12).

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APPENDIX: DERIVATION OF EQ. (20)

By making the change of variable $y=zx$ in the definition (20) of $G(x)$, this function is presented in the form

$$\begin{aligned} G(x) &= \int_x^\infty dy \frac{\ln(y)}{y} e^{-y} - \ln(x)E_1(x) \\ &= \int_1^\infty dy \frac{\ln(y)}{y} e^{-y} - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_1^x dy \ln(y) y^{n-1} \\ &\quad - \ln(x)E_1(x), \end{aligned} \tag{A1}$$

where in the second equality the integral from x to ∞ is written as a sum of two integrals—one from 1 to ∞ and the other from x to 1, and in the latter contribution the exponential function is replaced with its Taylor expansion. The first term in the sum is given by $\int_1^x dy \ln(y)/y=\frac{1}{2}\ln^2(x)$, and each subsequent term is given by $\int_1^x dy \ln(y)y^{n-1}=x^n \ln(x)/n+(1-x^n)/n^2$. Substitution of these results into the second part of Eq. (A1) and use of the representation (19) (second identity) for the exponential integral yields Eq. (20) with

$$\theta = \int_1^\infty dy \frac{\ln(y)}{y} e^{-y} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 n!} = 0.989\,048\,872\,2\ldots\ . \tag{A2}$$

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